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Closed-form sums for some perturbation series involving associated Laguerre polynomials

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Abstract

Infinite series $\sum_{n=1}^{\infty} \frac{(\alpha/2)_n}{n} \frac{1}{n!} {}_1F_1(-n, \gamma, x^2)$, where ${}_1F_1(-n, \gamma, x^2) = \frac{n!}{(\gamma)_n} L_n^{(\gamma-1)}(x^2)$, appear in the first-order perturbation correction for the wavefunction of the generalized spiked harmonic oscillator Hamiltonian $H = -\frac{d^2}{dx^2} + Bx^2 + \frac{A}{x^2} + \frac{\lambda}{x^\alpha}$, $0 \leqslant x < \infty$, $\alpha, \lambda > 0$, $A \geqslant 0$. It is proved that the series is convergent for all $x > 0$ and $2\gamma > \alpha$ where $\gamma = 1 + \frac{1}{2}\sqrt{1+4A}$. Closed-form sums are presented for these series for the cases $\alpha = 2$, 4 and 6. A general formula for finding the sum for $\frac{\alpha}{2} = 2 + m$, $m = 0, 1, 2, \ldots$ in terms of associated Laguerre polynomials is also provided.

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1. Introduction

Aguillera-Navarro and Guardiola [1] encounter some difficulties inherent in connection with attempts to derive the first-order perturbation expansion of the wavefunction of the spiked harmonic oscillator Hamiltonian

$$H = -\frac{d^2}{dx^2} + x^2 + \frac{\lambda}{x^\alpha} \qquad 0 \leqslant x < \infty \quad \alpha, \lambda > 0 \tag{1.1}$$

even for the case $\alpha = 2$, where a complete exact solution is also available. The reason for these difficulties lies in computing infinite series of the type

$$\sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n} \frac{1}{n!} {}_1F_1\left(-n; \tfrac{3}{2}; x^2\right) \tag{1.2}$$

where ${}_1F_1$ stands for the confluent hypergeometric function defined by

$${}_1F_1(-n; b; y) = \sum_{k=0}^{n} \frac{(-n)_k}{(b)_k} \frac{y^k}{k!} = \frac{n!}{(b)_n} L_n^{(b-1)}(y) \tag{1.3}$$

in terms of the associated Laguerre polynomials $L_n^{(b-1)}(y)$, and $(a)_n$ the shifted factorial (or *Pochhammer symbols*) defined by

$$(a)_0 = 1 \qquad (a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \qquad n = 1, 2, \ldots. \tag{1.4}$$

Recently, the present authors studied a more general Hamiltonian known now as the generalized spiked harmonic oscillator Hamiltonian [2–5]

$$H = H_0 + \lambda V = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + Bx^2 + \frac{A}{x^2} + \frac{\lambda}{x^\alpha} \qquad 0 \leqslant x < \infty \quad \alpha, \lambda > 0 \quad A \geqslant 0 \tag{1.5}$$

defined on the one-dimensional space $(0 \leqslant x < \infty)$ with eigenfunctions satisfying Dirichlet boundary conditions, that is to say, with wavefunctions vanishing at the boundaries. Herein equation (1.1) appears as a special case $(A = 0,\ B = 1)$. They found that the matrix elements of the operator $x^{-\alpha}$, with respect to the exact solutions of the Gol'dman and Krivchenkov Hamiltonian H_0, namely,

$$\psi_n(x) = (-1)^n \sqrt{\frac{2B^{\frac{\gamma}{2}}\Gamma(n+\gamma)}{n!\Gamma^2(\gamma)}} x^{\gamma-\frac{1}{2}} \,\mathrm{e}^{-\frac{\sqrt{B}}{2}x^2}\, {}_1F_1(-n, \gamma, \sqrt{B}x^2) \tag{1.6}$$

with exact eigenenergies

$$E_n = 2\sqrt{B}(2n+\gamma) \qquad n = 0, 1, 2, \ldots \qquad \gamma = 1 + \tfrac{1}{2}\sqrt{1+4A} \tag{1.7}$$

are given explicitly by the expressions

$$x_{mn}^{-\alpha} = (-1)^{n+m} B^{\frac{\alpha}{4}} \frac{\left(\frac{\alpha}{2}\right)_n}{(\gamma)_n} \frac{\Gamma(\gamma - \frac{\alpha}{2})}{\Gamma(\gamma)} \sqrt{\frac{(\gamma)_n(\gamma)_m}{n!m!}}\, {}_3F_2\left(-m, \gamma - \frac{\alpha}{2}, 1 - \frac{\alpha}{2}; \gamma, 1 - n - \frac{\alpha}{2}; 1\right) \tag{1.8}$$

and valid for all values of the parameters γ and α such that $\alpha < 2\gamma$. Furthermore, the matrix elements of the Hamiltonian (1.5) are given by

$$H_{mn} = \langle m|H|n\rangle \equiv 2\sqrt{B}(2n+\gamma)\delta_{nm} + \lambda(-1)^{m+n} B^{\frac{\alpha}{4}} \sqrt{\frac{(\gamma)_n(\gamma)_m}{n!m!}} \frac{\Gamma\left(\gamma - \frac{\alpha}{2}\right)\left(\frac{\alpha}{2}\right)_n}{(\gamma)_n\Gamma(\gamma)} \times {}_3F_2\left(-m, \gamma - \frac{\alpha}{2}, 1 - \frac{\alpha}{2}; \gamma, 1 - \frac{\alpha}{2} - n; 1\right). \tag{1.9}$$

Of particular interest are the elements

$$H_{0n} = \lambda(-1)^n B^{\frac{\alpha}{4}} \sqrt{\frac{(\gamma)_n}{n!}} \frac{\Gamma\left(\gamma - \frac{\alpha}{2}\right)}{\Gamma(\gamma)} \frac{\left(\frac{\alpha}{2}\right)_n}{(\gamma)_n} \qquad n \neq 0. \tag{1.10}$$

It is known that the first correction to the wavefunction by means of standard perturbation techniques leads to

$$\psi_0^{(1)}(x) = \sum_{n=1}^{\infty} \frac{H_{0n}}{E_0 - E_n} \psi_n(x) \tag{1.11}$$

where H_{0n} and $\psi_n(x)$ are given by equations (1.10) and (1.6), respectively. Thus, the first correction to the wavefunction of the Hamiltonian (1.5) is given by

$$\psi_0^{(1)}(x) = -\frac{B^{\frac{\alpha}{2}+\frac{\gamma}{4}-\frac{1}{2}}}{2\sqrt{2}} \frac{\Gamma\left(\gamma - \frac{\alpha}{2}\right)}{\Gamma(\gamma)\sqrt{\Gamma(\gamma)}} x^{\gamma-\frac{1}{2}} \,\mathrm{e}^{-\frac{\sqrt{B}}{2}x^2} \sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n} \frac{1}{n!}\, {}_1F_1(-n, \gamma, \sqrt{B}x^2). \tag{1.12}$$

The purpose of this paper is to find closed-form sums for the infinite series appearing in equation (1.12), namely,

$$\sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n} \frac{1}{n!} \, {}_1F_1(-n; \gamma; x^2) \tag{1.13}$$

where $2\gamma > \alpha$, $\alpha = 2, 4, 6, \ldots$ and we set $B = 1$, for simplicity. Because of equation (1.3), the results of this paper can be expressed equally well in terms of the associated Laguerre polynomials. The importance of closed-form sums for the infinite series (1.13) is that they help us to understand the abnormal behaviour of the standard, weak coupling, perturbation theory [1] for the singular Hamiltonians (1.1). Such infinite series were investigated earlier by the present authors [2], where they proved, in the case $\alpha < 2$, by means of the inverse Laplace transform, that

$$\sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n} \frac{1}{n!} \, {}_1F_1(-n, \gamma, x^2) = \frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \frac{\alpha}{2} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t \, t^{-\gamma} \left(1 - \frac{x^2}{t}\right)$$
$$\times \, {}_3F_2\left(1 + \frac{\alpha}{2}, 1, 1; 2, 2; 1 - \frac{x^2}{t}\right) \mathrm{d}t \qquad c > 0 \tag{1.14}$$

where $|1 - \frac{x^2}{t}| < 1$, which is indeed an important condition to insure the convergence of the series ${}_3F_2$ that appears on the right-hand side of (1.14). The functions ${}_3F_2$ and ${}_1F_1$, mentioned above, are special cases of the generalized hypergeometric function

$${}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} (\alpha_i)_k}{\prod_{j=1}^{q} (\beta_j)_k} \frac{z^k}{k!} \tag{1.15}$$

where p and q are non-negative integers and β_j $(j = 1, 2, \ldots, q)$ is such that it is not equal to zero or a negative integer. If the series does not terminate (one of α_i, $i = 1, 2, \ldots, p$, is a negative integer), then the series, in the case $p = q + 1$, converges or diverges according to whether $|z| < 1$ or $|z| > 1$. For $z = 1$, on the other hand, the series is convergent, provided $\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > 0$. This paper is organized as follows: in section 2 we demonstrate that the infinite series on the left-hand side of equation (1.14) converges for all $x > 0$ and $\gamma > \frac{\alpha}{2}$. Furthermore, the integral representation is still valid in such cases. In section 3 we prove that in the case $\alpha = 2$, we have

$$\sum_{n=1}^{\infty} \frac{(1)_n}{n \, n!} \, {}_1F_1(-n; \gamma; x^2) = \psi(\gamma) - \log x^2 \qquad \gamma > 1$$

while in the case $\alpha = 4$, we have

$$\sum_{n=1}^{\infty} \frac{(2)_n}{n \, n!} \, {}_1F_1(-n; \gamma; x^2) = \psi(\gamma) - \log x^2 + \frac{\gamma - 1}{x^2} - 1 \qquad \gamma > 2$$

and for the case $\alpha = 6$

$$\sum_{n=1}^{\infty} \frac{(3)_n}{n \, n!} \, {}_1F_1(-n; \gamma; x^2) = \psi(\gamma) - \log x^2 + \frac{\gamma - 1}{x^2} - \frac{3}{2} + \frac{(\gamma - 1)(\gamma - 2)}{2x^4} \qquad \gamma > 3.$$

In section 4 we prove our main result that for $\frac{\alpha}{2} = 2 + m$, $m = 0, 1, 2, \ldots$

$$\sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n \, n!} \, {}_1F_1(-n; \gamma; x^2) = \psi(\gamma) - \log x^2$$
$$- (m + 1) \sum_{k=0}^{m} \frac{(-m)_k}{(k + 1)^2} \left(-\frac{1}{x^2}\right)^k \left[L_k^{\gamma-1-k}(x^2) - \frac{(\gamma - 1)}{x^2} L_k^{\gamma-2-k}(x^2)\right]$$

where $L_n^{(a)}(\cdot)$ stands for the well-known associated Laguerre polynomials. An interpretation for the first-order correction of the wavefunction (1.12) as $x \to 0$ and some further remarks are given in section 5.

2. Integral representation and the convergence problem

In order to evaluate the sum in equation (1.13) for $\alpha > 0$ and $2\gamma > \alpha$, we require a suitable integral representation of the confluent hypergeometric function ${}_1F_1(-n, \gamma, x^2)$ over an appropriate contour, in order to interchange summation with integration and thereby readily conclude the absolute convergence of the series just mentioned. We find the inverse Laplace transform (integral) representation [6, p 116, formula (3)]

$$ {}_1F_1(a, \gamma, x^2) = \frac{\Gamma(\gamma)}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma} \left(1 - \frac{x^2}{t}\right)^{-a} \mathrm{d}t \tag{2.1} $$

under the conditions $\mathrm{Re}(\gamma) > 0$, $c > 0$, $\left|\arg\left(1 - \frac{x^2}{c}\right)\right| < \pi$ (which is clearly true for x real) to be the most advantageous for achieving this end.

Now turn to the evaluation of the summation in terms of the representation (2.1) written for $a = -n$, namely,

$$ {}_1F_1(-n, \gamma, x^2) = \frac{\Gamma(\gamma)}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma} \left(1 - \frac{x^2}{t}\right)^{n} \mathrm{d}t \qquad n = 0, 1, 2, \ldots \tag{2.2} $$

which substituted into the summation of equation (1.13) yields

$$ \begin{aligned} \sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n} \frac{1}{n!}\, {}_1F_1(-n, \gamma, x^2) &= (2\pi\mathrm{i})^{-1}\, \Gamma(\gamma) \sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n} \frac{1}{n!} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma} \left(1 - \frac{x^2}{t}\right)^{n} \mathrm{d}t \\ &= (2\pi)^{-1}\, \Gamma(\gamma) \sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n} \frac{1}{n!} \int_{-\infty}^{\infty} \mathrm{e}^{(c+\mathrm{i}y)} (c + \mathrm{i}y)^{-\gamma} \left(1 - \frac{x^2}{c + \mathrm{i}y}\right)^{n} \mathrm{d}y. \end{aligned} \tag{2.3} $$

The evaluation of this last infinite sum, involving integrations over the interval $(-\infty, \infty)$, is achieved by examining the summation of the integrand, namely,

$$ \sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n} \frac{1}{n!} \mathrm{e}^{(c+\mathrm{i}y)} (c + \mathrm{i}y)^{-\gamma} \left(1 - \frac{x^2}{c + \mathrm{i}y}\right)^{n} = \mathrm{e}^{(c+\mathrm{i}y)} (c + \mathrm{i}y)^{-\gamma} \sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n} \frac{1}{n!} \left(1 - \frac{x^2}{c + \mathrm{i}y}\right)^{n} \tag{2.4} $$

and demonstrating that it has an $L_1(-\infty, \infty)$-majorant. Hence, the existence of such a majorant will permit us to interchange summation with integration, as a result of the Lebesgue dominated convergence theorem. To arrive at such a majorant, we continue by noting that

$$ \begin{aligned} \sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n} \frac{1}{n!} \left(1 - \frac{x^2}{c + \mathrm{i}y}\right)^{n} &= \sum_{n=0}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_{n+1}}{n+1} \frac{1}{(n+1)!} \left(1 - \frac{x^2}{c + \mathrm{i}y}\right)^{n+1} \\ &= \frac{\alpha}{2} \left(1 - \frac{x^2}{c + \mathrm{i}y}\right) \sum_{n=0}^{\infty} \frac{\left(\frac{\alpha}{2} + 1\right)_n}{(2)_n} \frac{(1)_n}{(2)_n} \left(1 - \frac{x^2}{c + \mathrm{i}y}\right)^{n} \\ &= \frac{\alpha}{2} \left(1 - \frac{x^2}{c + \mathrm{i}y}\right) \sum_{n=0}^{\infty} \frac{\left(\frac{\alpha}{2} + 1\right)_n (1)_n (1)_n}{(2)_n (2)_n} \frac{\left(1 - \frac{x^2}{c+\mathrm{i}y}\right)^{n}}{n!} \\ &= \frac{\alpha}{2} \left(1 - \frac{x^2}{c + \mathrm{i}y}\right) {}_3F_2\left(\frac{\alpha}{2} + 1, 1, 1; 2, 2; 1 - \frac{x^2}{c + \mathrm{i}y}\right) \end{aligned} \tag{2.5} $$

as consequence of $(a)_{n+1} = a(a+1)_n$, $n! = (1)_n$ and $(n+1)! = (2)_n$. The series ${}_3F_2$, in equation (2.5), is convergent provided that $\left|1 - \frac{x^2}{c+\mathrm{i}y}\right| < 1$. Now, since

$$1 - \frac{x^2}{c+\mathrm{i}y} = 1 - \frac{x^2(c-\mathrm{i}y)}{c^2+y^2} = 1 - \frac{x^2c}{c^2+y^2} + \mathrm{i}\frac{x^2y}{c^2+y^2}$$

for which

$$\left|1 - \frac{x^2}{c+\mathrm{i}y}\right|^2 = 1 - \frac{x^2(2c-x^2)}{c^2+y^2} < 1$$

provided c is chosen large enough, i.e. $x^2 < 2c$. For such c we shall always have

$$0 < 1 - \frac{x^2(2c-x^2)}{c^2+y^2} < 1 \qquad \forall y \in R. \tag{2.6}$$

Furthermore, the series ${}_3F_2$, in equation (2.5), is absolutely convergent for $\left|1 - \frac{x^2}{c+\mathrm{i}y}\right| = 1$, provided that $\alpha < 2$ as a result of equation (1.15). We now return to the majorization of summation (2.4), which entails

$$\sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n\, n!} \mathrm{e}^{(c+\mathrm{i}y)}(c+\mathrm{i}y)^{-\gamma}\left(1 - \frac{x^2}{c+\mathrm{i}y}\right)^n = \mathrm{e}^{c+\mathrm{i}y}|c+\mathrm{i}y|^{-\gamma}\frac{\alpha}{2}\left(1 - \frac{x^2}{c+\mathrm{i}y}\right)$$
$$\times {}_3F_2\left(\frac{\alpha}{2}+1, 1, 1; 2, 2; 1 - \frac{x^2}{c+\mathrm{i}y}\right) < A(\alpha, c)|c+\mathrm{i}y|^{-\gamma} \tag{2.7}$$

where the convergence of ${}_3F_2$ and also the condition $\left|1 - \frac{x^2}{c+\mathrm{i}y}\right| < 1$ were made use of. The most important aspect of inequality (2.7) is the appearance of the $L_1(-\infty, \infty)$-function $|c+\mathrm{i}y|^{-\gamma}$ of variable y majorizing the series

$$\sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n\, n!}\left|\mathrm{e}^{\sqrt{B}(c+\mathrm{i}y)}(c+\mathrm{i}y)^{-\gamma}\left(1 - \frac{x^2}{c+\mathrm{i}y}\right)^n\right|$$

and this aspect justifies the evaluation of summation (2.4) by means of the Lebesgue dominated convergence theorem. Thus we specifically have

$$\sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n}\frac{1}{n!}\,{}_1F_1(-n,\gamma,x^2) = \frac{\Gamma(\gamma)}{2\pi\mathrm{i}}\sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n}\frac{1}{n!}\int_{-\infty}^{\infty} \mathrm{e}^{(c+\mathrm{i}y)}(c+\mathrm{i}y)^{-\gamma}\left(1 - \frac{x^2}{c+\mathrm{i}y}\right)^n \mathrm{i}\,\mathrm{d}y$$
$$= \frac{\Gamma(\gamma)}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \mathrm{e}^{(c+\mathrm{i}y)}(c+\mathrm{i}y)^{-\gamma}\left[\sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n}\frac{1}{n!}\left(1 - \frac{x^2}{c+\mathrm{i}y}\right)^n\right]\mathrm{i}\,\mathrm{d}y$$
$$= \frac{\Gamma(\gamma)}{2\pi}\frac{\alpha}{2}\int_{-\infty}^{\infty} \mathrm{e}^{(c+\mathrm{i}y)}(c+\mathrm{i}y)^{-\gamma}\left(1 - \frac{x^2}{c+\mathrm{i}y}\right)$$
$$\times {}_3F_2\left(1, 1, 1+\frac{\alpha}{2}; 2, 2; 1 - \frac{x^2}{c+\mathrm{i}y}\right)\mathrm{d}y \tag{2.8}$$

which is an effective straightforward and precise determination of the summation $\sum_{n=1}^{\infty} \frac{(\alpha/2)_n}{n}\frac{1}{n!}\,{}_1F_1(-n,\gamma,x^2)$ in terms of integrals of higher order hypergeometric function for arbitrary $\alpha < 2\gamma$. However, by utilizing $t = c + \mathrm{i}y$ we reconvert the last expression of relation (2.8) to the inverse Laplace transform format, namely,

$$\sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n}\frac{1}{n!}\,{}_1F_1(-n,\gamma,x^2) = \frac{\Gamma(\gamma)}{2\pi\mathrm{i}}\frac{\alpha}{2}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t t^{-\gamma}\left(1 - \frac{x^2}{t}\right)$$
$$\times {}_3F_2\left(1, 1, 1+\frac{\alpha}{2}; 2, 2; 1 - \frac{x^2}{t}\right)\mathrm{d}t \tag{2.9}$$

valid for all $\alpha < 2\gamma$. The computation of this expression is carried out in the next section.

3. Closed-form sums

Lemma 1. *For* $\gamma > 1$

$$\sum_{n=1}^{\infty} \frac{1}{n}\, {}_1F_1(-n, \gamma, x^2) = \psi(\gamma) - \log x^2. \tag{3.1}$$

Proof. For $\alpha = 2$ and $(1)_n = n!$, equation (2.9) leads to

$$\sum_{n=1}^{\infty} \frac{1}{n}\, {}_1F_1(-n, \gamma, x^2) = \frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma} \left(1 - \frac{x^2}{t}\right) {}_3F_2 \left(2, 1, 1; 2, 2; 1 - \frac{x^2}{t}\right) \mathrm{d}t$$

$$= \frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma} \left(1 - \frac{x^2}{t}\right) {}_2F_1 \left(1, 1; 2; 1 - \frac{x^2}{t}\right) \mathrm{d}t$$

It is known, however, that

$${}_2F_1(1, 1; 2; z) = -\tfrac{1}{z} \log(1 - z) \qquad |z| < 1$$

Thus, for $z = 1 - \frac{x^2}{t}$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n}\, {}_1F_1(-n, \gamma, x^2) = -\frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma} \log\left(\frac{x^2}{t}\right) \mathrm{d}t$$

$$= -\log x^2\, \frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma}\, \mathrm{d}t + \frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma} \log t\, \mathrm{d}t.$$

The first integral on the right-hand side can be computed by means of the reciprocal of the Γ-function [6, p 17, formula (5)] or by means of the inverse Laplace transform of $f(t) = t^{-\gamma}$ for $\gamma > 0$

$$[\Gamma(\gamma)]^{-1} = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma}\, \mathrm{d}t \qquad c > 0 \quad \gamma > 0. \tag{3.2}$$

Further, by differentiating equation (3.2) with respect to γ, we get

$$\frac{\psi(\gamma)}{\Gamma(\gamma)} = \frac{\Gamma'(\gamma)}{[\Gamma(\gamma)]^2} = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma} \log(t)\, \mathrm{d}t \qquad c > 0 \quad \gamma > 0 \tag{3.3}$$

where $\psi(\gamma)$ is the digamma function defined as $\psi(\gamma) = \frac{\mathrm{d}}{\mathrm{d}\gamma} \log \Gamma(\gamma)$. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n}\, {}_1F_1(-n, \gamma, x^2) = \psi(\gamma) - \log(x^2) \qquad \text{for} \quad \gamma > 1$$

as required. □

The result of lemma 1 is not new indeed, and it was proved earlier by Toscano [7] by means of extensive use of calculus of finite difference. Toscano's result [7], however, was given in terms of associated Laguerre polynomials $L_n^{(\gamma)}(\cdot)$ where he proved that

$$\sum_{n=1}^{\infty} \frac{(n-1)!}{\Gamma(n+\gamma)} L_n^{(\gamma-1)}(y) = \frac{1}{\Gamma(\gamma)} [\psi(\gamma) - \log y]. \tag{3.4}$$

For comparison, we use the relation between the confluent hypergeometric function ${}_1F_1(-n; \gamma + 1; \cdot)$ and the associated Laguerre polynomials $L_n^{(\gamma)}(\cdot)$, namely,

$${}_1F_1(-n, \gamma + 1, \cdot) = \frac{\Gamma(n+1)\Gamma(\gamma+1)}{\Gamma(n+\gamma+1)} L_n^{(\gamma)}(\cdot). \tag{3.5}$$

Thus,

$$\sum_{n=1}^{\infty}\frac{1}{n}\,{}_1F_1(-n,\gamma,x^2)=\Gamma(\gamma)\sum_{n=1}^{\infty}\frac{(n-1)!}{\Gamma(n+\gamma)}L_n^{(\gamma-1)}(x^2) \tag{3.6}$$

and this leads to the same results as lemma 1. In other words, lemma 1 gives an independent proof of Toscano's result [7].

In order to find closed sums for equation (2.9) for positive even numbers of α, we start with the reduction formula for ${}_3F_2(a,b,1;c,2;z)$ as given by Luke [6, p 111, formula (40)]:

$$z\,{}_3F_2(a,b,1;c,2;z)=\frac{(c-1)}{(a-1)(b-1)}\left[{}_2F_1(a-1,b-1;c-1;z)-1\right]\qquad |z|<1. \tag{3.7}$$

The purpose of the following lemma is to find the limit of Luke's identity as $b\to 1$.

Lemma 2. *For $a\neq 1$, $c\neq 1$ and $\left|\frac{z}{z-1}\right|<1$,*

$$z\,{}_3F_2(a,1,1;c,2;z)=\frac{(c-1)}{(a-1)}\left[\frac{(c-a)}{(c-1)}\left(\frac{z}{z-1}\right)\right.$$
$$\left.\times\,{}_3F_2\left(c-a+1,1,1;c,2;\frac{z}{z-1}\right)-\log(1-z)\right]. \tag{3.8}$$

Proof. From Pfaff's transformation [8] for ${}_2F_1$,

$${}_2F_1(a,b;c;z)=(1-z)^{-a}\,{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right)\qquad\left|\frac{z}{z-1}\right|<1 \tag{3.9}$$

which is also known as Euler's second identity, we have, by means of equation (2.9), that

$$z\,{}_3F_2(a,1,1;c,2;z)=\lim_{b\to 1}\frac{(c-1)}{(a-1)(b-1)}$$
$$\times\left[(1-z)^{-(b-1)}\,{}_2F_1\left(c-a,b-1;c-1;\frac{z}{z-1}\right)-1\right].$$

Using the identity

$$(1-z)^{-(b-1)}=\exp[-(b-1)\log(1-z)]$$

and the series representation

$${}_2F_1\left(c-a,b-1;c-1;\frac{z}{z-1}\right)=\sum_{n=0}^{\infty}\frac{(c-a)_n(b-1)_n}{(c-1)_n\,n!}\left(\frac{z}{z-1}\right)^n$$
$$=1+\sum_{n=1}^{\infty}\frac{(c-a)_n(b-1)_n}{(c-1)_n\,n!}\left(\frac{z}{z-1}\right)^n$$

we have

$$z\,{}_3F_2(a,1,1;c,2;z)=\lim_{b\to 1}\frac{(c-1)}{(a-1)(b-1)}$$
$$\times\left[\left\{1-(b-1)\log(1-z)+\tfrac{1}{2}(b-1)^2[\log(1-z)]^2+\mathrm{O}(b-1)^3\right\}\right.$$
$$\left.\times\left\{1+\sum_{n=1}^{\infty}\frac{(c-a)_n(b-1)_n}{(c-1)_n\,n!}\left(\frac{z}{z-1}\right)^n\right\}-1\right].$$

Further, since

$$\sum_{n=1}^{\infty} \frac{(c-a)_n (b-1)_n}{(c-1)_n n!} \left(\frac{z}{z-1}\right)^n = \sum_{n=0}^{\infty} \frac{(c-a)_{n+1} (b-1)_{n+1}}{(c-1)_{n+1} (n+1)!} \left(\frac{z}{z-1}\right)^{n+1}$$
$$= \frac{(c-a)(b-1)}{(c-1)} \left(\frac{z}{z-1}\right) \sum_{n=0}^{\infty} \frac{(c-a+1)_n (b)_n (1)_n}{(c)_n (2)_n n!} \left(\frac{z}{z-1}\right)^n$$
$$= \frac{(c-a)(b-1)}{(c-1)} \left(\frac{z}{z-1}\right) {}_3F_2\left(c-a+1, b, 1; c, 2; \frac{z}{z-1}\right)$$

where we implement $(a)_{n+1} = a(a+1)_n$ and the series representation of ${}_3F_2$ by means of equation (1.15). Thus we have

$$z\,{}_3F_2(a,1,1;c,2;z) = \lim_{b\to 1} \frac{(c-1)}{(a-1)(b-1)} \left[\frac{(c-a)(b-1)}{(c-1)} \left(\frac{z}{z-1}\right)\right.$$
$$\times\, {}_3F_2\left(c-a+1, b, 1; c, 2; \frac{z}{z-1}\right) - (b-1)\log(1-z)$$
$$-\frac{(c-a)(b-1)^2}{(c-1)} \left(\frac{z}{z-1}\right) \log(1-z)$$
$$\left.\times\, {}_3F_2\left(2-\frac{\alpha}{2}, b, 1; 2, 2; \frac{z}{z-1}\right) + \mathrm{O}(b-1)^2\right]$$
$$= \frac{(c-1)}{(a-1)} \left[\frac{(c-a)}{(c-1)} \left(\frac{z}{z-1}\right) {}_3F_2\left(c-a+1, 1, 1; c, 2; \frac{z}{z-1}\right)\right.$$
$$\left. - \log(1-z)\right].$$

This proves the lemma. □

As a direct application of this lemma, we have for $a = 1 + \frac{\alpha}{2}$ and $c = 2$,

$$z\,{}_3F_2\left(1+\frac{\alpha}{2}, 1, 1; 2, 2; z\right) = \frac{2}{\alpha}\left[\left(1-\frac{\alpha}{2}\right)\frac{z}{z-1}\,{}_3F_2\left(c-a+1, 1; 2, 2; \frac{z}{z-1}\right)\right.$$
$$\left. - \log(1-z)\right] \qquad \left|\frac{z}{z-1}\right| < 1. \tag{3.10}$$

For the purpose of our applications, where we have $z = 1 - \frac{x^2}{t}$ for $t = c + \mathrm{i}y$, we must note

$$\left|\frac{z}{z-1}\right|^2 = \left(\frac{z}{z-1}\right)\left(\frac{\bar{z}}{\bar{z}-1}\right) < 1$$

which leads to $\Re(z) < \frac{1}{2}$. However, the real part of $z = 1 - \frac{x^2}{c+\mathrm{i}y} = 1 - \frac{x^2(c-\mathrm{i}y)}{c^2+y^2}$ is

$$1 - \frac{x^2 c}{c^2+y^2} < \frac{1}{2} \to \frac{1}{2} < \frac{x^2 c}{c^2+y^2} < \frac{x^2}{c}$$

that is to say $\frac{c}{2} < x^2$, which does not contradict our requirement as given by equation (2.5). Therefore, lemma 2 can be used with arbitrary values of α provided that $\alpha < 2\gamma$. However, for $2 - \frac{\alpha}{2} = -m$, $\quad m = 0, 1, 2, \ldots$, the series ${}_3F_2$ on the right-hand side of equation (3.10) terminates and the convergence problem does not arise. In this case we have:

Lemma 3. *For $2 - \frac{\alpha}{2} = -m,\ m = 0, 1, 2, \ldots,$ we have*

$$\sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n} \frac{1}{n!}\, {}_1F_1(-n, \gamma, x^2) = \left(1 - \frac{\alpha}{2}\right) \frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma} \left(1 - \frac{t}{x^2}\right) \times {}_3F_2\left(2 - \frac{\alpha}{2}, 1, 1; 2, 2; 1 - \frac{t}{x^2}\right) \mathrm{d}t + \psi(\gamma) - \log x^2. \tag{3.11}$$

Proof. Using lemma 2 and the fact that $z = 1 - \frac{x^2}{t}$, equation (2.9) leads to

$$\sum_{n=1}^{\infty} \frac{\left(\frac{\alpha}{2}\right)_n}{n} \frac{1}{n!}\, {}_1F_1(-n, \gamma, x^2) = \left(1 - \frac{\alpha}{2}\right) \frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma} \left(1 - \frac{t}{x^2}\right) \times {}_3F_2\left(2 - \frac{\alpha}{2}, 1, 1; 2, 2; 1 - \frac{t}{x^2}\right) \mathrm{d}t - \frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma} \log\left(\frac{x^2}{t}\right) \mathrm{d}t. \tag{3.12}$$

The second integral on the right-hand side of equation (3.12) is already computed by means of lemma 1 and leads to

$$\frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^t t^{-\gamma} \log\left(\frac{x^2}{t}\right) \mathrm{d}t = \log x^2 - \psi(\gamma)$$

which completes the proof of the lemma. □

3.1. The case $\alpha = 4$

In this case $2 - \frac{\alpha}{2} = 0$ and lemma 3 leads to

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(2)_n}{n\, n!}\, {}_1F_1(-n, \gamma, x^2) &= -\frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma} \left(1 - \frac{t}{x^2}\right) \mathrm{d}t + \psi(\gamma) - \log x^2 \\ &= -\frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma} \mathrm{d}t + \frac{1}{x^2} \frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma+1} \mathrm{d}t + \psi(\gamma) - \log x^2 \\ &= \frac{\gamma - 1}{x^2} - 1 + \psi(\gamma) - \log x^2 \qquad \gamma > 2 \end{aligned} \tag{3.1.1}$$

where we invoke equation (3.2). There is, indeed, an independent confirmation for this result. Since $(2)_n = (1 + n)(1)_n$, the infinite sum in (1.13), reads

$$\sum_{n=1}^{\infty} \frac{1 + n}{n}\, {}_1F_1(-n; \gamma; x^2) = \sum_{n=1}^{\infty} \frac{1}{n}\, {}_1F_1(-n; \gamma; x^2) + \sum_{n=1}^{\infty} {}_1F_1(-n; \gamma; x^2). \tag{3.1.2}$$

The first series on the right-hand side is summable by means of Toscano's result [7] (regardless of the integral representation). For the second sum on the right-hand side, we refer to Buchholz's identity [9],

$$\sum_{n=0}^{\infty} \frac{(-\nu)_n \Gamma(\gamma + \nu + 1)}{\Gamma(n + \gamma + 1)} L_n^{(\gamma)}(y) = y^{\nu} \qquad \gamma + \nu > -1 \quad \nu \neq 0, 1, 2, \ldots. \tag{3.1.3}$$

Using equation (3.5), we have

$$\sum_{n=0}^{\infty} \frac{(-\nu)_n \Gamma(\gamma + \nu + 1)}{n! \Gamma(\gamma + 1)}\, {}_1F_1(-n; \gamma + 1; y) = y^{\nu} \qquad \gamma + \nu > -1. \tag{3.1.4}$$

Setting $\nu = -1$, we get

$$\sum_{n=0}^{\infty} {}_1F_1(-n; \gamma+1; y) = \gamma y^{-1} \qquad \gamma > 0$$

or

$$\sum_{n=1}^{\infty} {}_1F_1(-n; \gamma; x^2) = \frac{\gamma-1}{x^2} - 1 \qquad \gamma > 1 \tag{3.1.5}$$

and thus

$$\sum_{n=1}^{\infty} \frac{(2)_n}{n} \frac{1}{n!} {}_1F_1(-n; \gamma; x^2) = \psi(\gamma) - \log x^2 + \frac{\gamma-1}{x^2} - 1$$

which confirms our result as given by equation (3.1.1).

3.2. The case $\alpha = 6$

By means of equation (3.10), we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(3)_n}{n\, n!} {}_1F_1(-n, \gamma, x^2)
&= -2\frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma} \left(1 - \frac{t}{x^2}\right)\left(1 - \frac{1}{4}\left(1 - \frac{t}{x^2}\right)\right) \mathrm{d}t + \psi(\gamma) - \log x^2 \\
&= \frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma} \left(-\frac{3}{2} + \frac{t}{x^2} + \frac{t^2}{2x^4}\right) \mathrm{d}t + \psi(\gamma) - \log x^2 \\
&= -\frac{3}{2}\frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma}\, \mathrm{d}t + \frac{1}{x^2}\frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma+1}\, \mathrm{d}t \\
&\quad + \frac{1}{2x^4}\frac{\Gamma(\gamma)}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^t\, t^{-\gamma+2}\, \mathrm{d}t + \psi(\gamma) - \log x^2 \\
&= -\frac{3}{2} + \frac{1}{x^2}\frac{\Gamma(\gamma)}{\Gamma(\gamma-1)} + \frac{1}{2x^4}\frac{\Gamma(\gamma)}{\Gamma(\gamma-2)} + \psi(\gamma) - \log x^2 \\
&= -\frac{3}{2} + \frac{\gamma-1}{x^2} + \frac{(\gamma-1)(\gamma-2)}{2x^4} + \psi(\gamma) - \log x^2 \qquad \gamma > 3
\end{aligned} \tag{3.2.1}$$

where we invoke equation (3.2). These results can also be confirmed by an independent proof. Since $(3)_n = \frac{1}{2}(n^2 + 3n + 2)(1)_n$, the infinite series (1.13) becomes in this case

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(3)_n}{n} \frac{1}{n!} {}_1F_1(-n; \gamma; x^2) &= \frac{1}{2} \sum_{n=1}^{\infty} n\, {}_1F_1(-n; \gamma; x^2) \\
&\quad + \frac{3}{2} \sum_{n=1}^{\infty} {}_1F_1(-n; \gamma; x^2) + \sum_{n=1}^{\infty} \frac{1}{n} {}_1F_1(-n; \gamma; x^2).
\end{aligned}$$

The second and third series on the right-hand side are summable by means of equations (3.1.5) and (3.6), respectively, regardless of the integral representation. For the first series on the right-hand side, it is enough to take $\nu = -2$ in equation (3.1.4) to conclude that

$$\sum_{n=1}^{\infty} n\, {}_1F_1(-n; \gamma; x^2) = \frac{(\gamma-1)(\gamma-2)}{x^4} - \frac{\gamma-1}{x^2} \qquad \gamma > 2. \tag{3.2.2}$$

This leads to

$$\sum_{n=1}^{\infty}\frac{(3)_n}{n}\frac{1}{n!}\,{}_1F_1(-n;\gamma;x^2)=\frac{\gamma-1}{x^2}-\frac{3}{2}+\frac{(\gamma-1)(\gamma-2)}{2x^4}-\log x^2+\psi(\gamma)\qquad \gamma>3.$$

4. General case

The results just mentioned for $\alpha=4$ and $\alpha=6$ can be generalized indeed to any α such that $2-\frac{\alpha}{2}=-m, m=0,1,2,\ldots$.

Lemma 4. *For* $2-\frac{\alpha}{2}=-m,\ m=0,1,2,\ldots$ *and* $\gamma>\frac{\alpha}{2}$, *we have*

$$\sum_{n=1}^{\infty}\frac{\left(\frac{\alpha}{2}\right)_n}{n\,n!}\,{}_1F_1(-n;\gamma;x^2)=\psi(\gamma)-\log x^2-(m+1)\sum_{k=0}^{m}\frac{(-m)_k(1)_k}{(2)_k(2)_k}\times\left[{}_2F_0\left(-k,1-\gamma;-;-\frac{1}{x^2}\right)-\frac{\gamma-1}{x^2}\,{}_2F_0\left(-k,2-\gamma;-;-\frac{1}{x^2}\right)\right].\qquad(4.1)$$

Proof. Using lemma 3, we have for $2-\frac{\alpha}{2}=-m,\ m=0,1,2,\ldots,$

$$\sum_{n=1}^{\infty}\frac{\left(\frac{\alpha}{2}\right)_n}{n}\frac{1}{n!}\,{}_1F_1(-n,\gamma,x^2)=\psi(\gamma)-\log x^2-(m+1)\frac{\Gamma(\gamma)}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{e}^t\,t^{-\gamma}\left(1-\frac{t}{x^2}\right)\times{}_3F_2\left(-m,1,1;2,2;1-\frac{t}{x^2}\right)\mathrm{d}t.$$

The function ${}_3F_2\big(-m,1,1;2,2;1-\frac{t}{x^2}\big)$ is a terminated series, specifically a polynomial of degree m, and therefore we may integrate term by term using the series representation of ${}_3F_2$. We have

$$\begin{aligned}I_m^{\gamma}(x)&=-(m+1)\frac{\Gamma(\gamma)}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{e}^t\,t^{-\gamma}\left(1-\frac{t}{x^2}\right){}_3F_2\left(-m,1,1;2,2;1-\frac{t}{x^2}\right)\mathrm{d}t\\&=-(m+1)\frac{\Gamma(\gamma)}{2\pi\mathrm{i}}\sum_{k=0}^{m}\frac{(-m)_k(1)_k}{(2)_k(2)_k}\left[\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{e}^t\,t^{-\gamma}\left(1-\frac{t}{x^2}\right)^k\mathrm{d}t\right.\\&\quad\left.-\frac{1}{x^2}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{e}^t\,t^{-\gamma+1}\left(1-\frac{t}{x^2}\right)^k\mathrm{d}t\right].\end{aligned}$$

Since

$$\left(1-\frac{t}{x^2}\right)^k=\sum_{l=0}^{k}\frac{(-k)_l}{l!}\left(\frac{t}{x^2}\right)^l,\ \text{finite number of terms,}$$

we have

$$\begin{aligned}I_m^{\gamma}(x)&=-(m+1)\sum_{k=0}^{m}\frac{(-m)_k(1)_k}{(2)_k(2)_k}\left[\sum_{l=0}^{k}\frac{(-k)_l}{l!}\frac{1}{x^{2l}}\frac{\Gamma(\gamma)}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{e}^t\,t^{-\gamma+l}\,\mathrm{d}t\right.\\&\quad\left.-\sum_{l=0}^{k}\frac{(-k)_l}{l!}\frac{1}{x^{2l+2}}\frac{\Gamma(\gamma)}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{e}^t\,t^{-\gamma+l+1}\,\mathrm{d}t\right]\\&=-(m+1)\sum_{k=0}^{m}\frac{(-m)_k(1)_k}{(2)_k(2)_k}\left[\sum_{l=0}^{k}\frac{(-k)_l}{l!}\frac{1}{x^{2l}}\frac{\Gamma(\gamma)}{\Gamma(\gamma-l)}\right.\\&\quad\left.-\sum_{l=0}^{k}\frac{(-k)_l}{l!}\frac{1}{x^{2l+2}}\frac{\Gamma(\gamma)}{\Gamma(\gamma-l-1)}\right]\end{aligned}$$

where we have used equation (3.2) for $\gamma > l + 1$. From the identity $\Gamma(\gamma - l) = \Gamma(\gamma)(\gamma)_{-l} = \Gamma(\gamma)\frac{(-1)^l}{(1-\gamma)_l}$ and $\Gamma(\gamma - l - 1) = \Gamma(\gamma - 1)(\gamma - 1)_{-l} = \Gamma(\gamma - 1)\frac{(-1)^l}{(2-\gamma)_l}$, we have now

$$I_m^\gamma(x) = -(m+1)\sum_{k=0}^{m}\frac{(-m)_k(1)_k}{(2)_k(2)_k}\left[\sum_{l=0}^{k}\frac{(-k)_l(1-\gamma)_l}{l!}\left(-\frac{1}{x^2}\right)^l - \frac{(\gamma-1)}{x^2}\sum_{l=0}^{k}\frac{(-k)_l(2-\gamma)_l}{l!}\left(-\frac{1}{x^2}\right)^l\right]$$

which finally leads to

$$I_m^\gamma(x) = -(m+1)\sum_{k=0}^{m}\frac{(-m)_k(1)_k}{(2)_k(2)_k}\left[{}_2F_0\left(-k, 1-\gamma; -; -\frac{1}{x^2}\right) - \frac{(\gamma-1)}{x^2}{}_2F_0\left(-k, 2-\gamma; -; -\frac{1}{x^2}\right)\right]$$

by means of equation (1.15). □

The significance of this lemma is that the infinite series of equation (2.9) can now be replaced by a finite series that is much easier to calculate. To illustrate the use of this lemma, we shall now find the infinite series of equation (2.9) for the case $\alpha = 8$, i.e. $m = 2$, since

$$\sum_{n=1}^{\infty}\frac{(4)_n}{n\,n!}\,{}_1F_1(-n;\gamma;x^2) = \psi(\gamma) - \log x^2 - 3\sum_{k=0}^{2}\frac{(-m)_k(1)_k}{(2)_k(2)_k} \times\left[{}_2F_0\left(-k, 1-\gamma; -; -\frac{1}{x^2}\right) - \frac{\gamma-1}{x^2}{}_2F_0\left(-k, 2-\gamma; -; -\frac{1}{x^2}\right)\right]$$

and since

$${}_2F_0\left(0, 1-\gamma; -; -\frac{1}{x^2}\right) = 1$$

$${}_2F_0\left(-1, 1-\gamma; -; -\frac{1}{x^2}\right) = 1 - \frac{(\gamma-1)}{x^2}$$

$${}_2F_0\left(-2, 1-\gamma; -; -\frac{1}{x^2}\right) = 1 - \frac{2(\gamma-1)}{x^2} + \frac{(\gamma-1)(\gamma-2)}{x^4}$$

and similarly for ${}_2F_0\big(-k, 2-\gamma; -; -\frac{1}{x^2}\big)$, $k = 0, 1, 2$. It is a straightforward calculation to find a closed-form sum for the infinite series (2.9) for $\alpha = 8$ which leads in this case to

$$\sum_{n=1}^{\infty}\frac{(4)_n}{n\,n!}\,{}_1F_1(-n;\gamma;x^2) = \psi(\gamma) - \log x^2 - \frac{11}{6} + \frac{(\gamma-1)(\gamma-2)(\gamma-3)}{3x^6} + \frac{(\gamma-1)(\gamma-2)}{2x^4} + \frac{(\gamma-1)}{x^2} \tag{4.2}$$

valid for $\gamma > 4$. It is interesting to mention here that the result of lemma 4 can be written in terms of the well-known associated Laguerre polynomials. Indeed, from the identity [10]

$$(-1)^n L_k^{a-n}(y) = \frac{y^n}{n!}\,{}_2F_0\left(-n, -a; -; -\frac{1}{y}\right)$$

the result of lemma 4 can be written as

$$\sum_{n=1}^{\infty}\frac{\left(\frac{\alpha}{2}\right)_n}{n\,n!}\,{}_1F_1(-n;\gamma;x^2)=\psi(\gamma)-\log x^2-(m+1)\sum_{k=0}^{m}\frac{(-m)_k}{(k+1)^2}\left(-\frac{1}{x^2}\right)^k \times\left[L_k^{\gamma-1-k}(x^2)-\frac{(\gamma-1)}{x^2}L_k^{\gamma-2-k}(x^2)\right] \quad (4.3)$$

for $\frac{\alpha}{2}=2+m \quad m=0,1,2,\ldots.$

5. Concluding remarks

It is important to note that for $x=0$, the infinite series (1.13) for $\alpha\geqslant 2$ indeed diverges. This follows from the fact that ${}_1F_1(-n;\gamma;0)=1$ and

$$\sum_{n=1}^{\infty}\frac{\left(\frac{\alpha}{2}\right)_n}{n}\frac{1}{n!}=\frac{\alpha}{2}\sum_{n=0}^{\infty}\frac{(1+\frac{\alpha}{2})_n(1)_n}{(2)_n}\frac{(1)_n}{(2)_n\,n!}=\frac{\alpha}{2}{}_3F_2\left(1+\frac{\alpha}{2},1,1;2,2;1\right)$$

which is absolutely convergent for $\alpha<2$. Therefore, for our results concerning $\alpha=2,4,\ldots$ and for the integral representation (2.9) in general, we must consider $x>0$. The divergence of the infinite series in the expression of the first-order perturbation correction of the wavefunction (1.12) as $x\to 0$ is indeed controlled by the coefficient term $x^{\gamma-1/2}$ as well as by the coefficient $\mathrm{e}^{-x^2/2}$ for $x\to\infty$. To illustrate the point further, we consider the case $\alpha=2$. In this case the infinite series in equation (1.12) is summable by means of lemma 1 and the first-order perturbation correction now reads

$$\psi_0^{(1)}(x)=\frac{1}{\sqrt{2}}\frac{1}{(\gamma-1)\sqrt{\Gamma(\gamma)}}x^{\gamma-\frac{1}{2}}\,\mathrm{e}^{-\frac{x^2}{2}}\left[\log x-\tfrac{1}{2}\psi(\gamma)\right] \qquad \gamma>1. \quad (5.1)$$

Since $\lim_{x\to 0}x^{\gamma-\frac{1}{2}}\log x=0$ for $\gamma>1$, we have $\psi_0^{(1)}(0)=0$. Consequently, the closed-form sums of the infinite series (1.13) contribute for intermediate values $0<x<\infty$ of the wavefunction rather than the boundaries.

The question posed by Aguillera-Navarro and Guardiola [1] concerning a special summation formula for equation (1.2) in the case $\alpha=2$ can now be answered with the aid of lemma 1, which leads to

$$\sum_{n=1}^{\infty}\frac{1}{n}\,{}_1F_1\left(-n;\tfrac{3}{2};x^2\right)=\psi\left(\tfrac{3}{2}\right)-\log x^2 \quad (5.2)$$

and the first-order perturbation correction is given by means of equation (5.1) as

$$\psi_0^{(1)}(x)=2\pi^{-1/4}x\,\mathrm{e}^{-\frac{x^2}{2}}\left[\log x-\tfrac{1}{2}\psi\left(\tfrac{3}{2}\right)\right] \quad (5.3)$$

which matches the first-order perturbation correction expansion, in powers of λ, of the exact wavefunction $\psi_0(x)$, that is equation (1.6).

The condition $\gamma>\frac{\alpha}{2}, \quad \alpha=2,4,6,\ldots$ imposed on the closed-form sums is too strong, for they are indeed valid for weaker conditions. For example,

$$\sum_{n=1}^{\infty}\frac{1}{n}\,{}_1F_1(-n,\gamma,x^2)=\psi(\gamma)-\log x^2 \qquad \text{valid for all } \gamma>0$$

$$\sum_{n=1}^{\infty}\frac{n+1}{n}\,{}_1F_1(-n,\gamma,x^2)=\frac{\gamma-1}{x^2}-1+\psi(\gamma)-\log x^2 \qquad \text{valid for all } \gamma>1$$

$$\sum_{n=1}^{\infty} \frac{\frac{1}{2}(n^2+3n+2)}{n}\, {}_1F_1(-n,\gamma,x^2) = -\frac{3}{2} + \frac{\gamma-1}{x^2} + \frac{(\gamma-1)(\gamma-2)}{2x^4} + \psi(\gamma) - \log x^2 \qquad \text{valid for all } \gamma > 2.$$

However, the condition has been imposed in order to meet the matrix elements' convergence requirements.

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References

[1] Aguilera-Navarro V C and Guardiola R 1991 *J. Math. Phys.* **32** 2135
[2] Hall R, Saad N and von Keviczky A 2001 *J. Phys. A: Math. Gen.* **34** 1169
[3] Hall R, Saad N and von Keviczky A 1998 *J. Math. Phys.* **39** 6345
[4] Hall R and Saad N 2000 *J. Phys. A: Math. Gen.* **33** 569
[5] Hall R and Saad N 2000 *J. Phys. A: Math. Gen.* **33** 5531
[6] Luke Y L 1969 *The Special Functions and Their Approximation* (New York: Academic)
[7] Nota di Letterio Toscano 1949 *Boll. Un. Mat. Ital.* **3** 398
[8] Andrews G E, Askey R and Roy R 1999 *Special Functions* (Cambridge: Cambridge University Press) p 68 theorem 2.2.5
[9] Buchholz H 1969 *The Confluent Hypergeometric Function* (Berlin: Springer) p 144
[10] McBride E B 1971 *Obtaining Generating Functions* (Berlin: Springer) p 68